VACATION WORK FOR AS MATHEMATICS

Before starting any Mathematics AS course at Waddesdon you will need to have studied the following at GCSE:

- Solving linear and quadratic simultaneous equations
- Expanding brackets – single, double and triple
- Factorising
- Solving linear equations
- Solve Quadratic equation – by factorising, using the quadratic formula and completing the square and locate the position of the turning point
- Solving linear and quadratic inequalities
- Indices
- Surds
- Co-ordinate Geometry: $y = mx+c$, find the mid-point, distance between two points
- Functions
- Simple algebraic Proof

Each of these topics is essential background knowledge for the courses, which you will follow in September.

You will be expected to have taken the Higher Tier examination at GCSE and achieved a minimum of a grade 6, but preferably a 7 or better. In addition, you will be expected to pass an algebra test based on GCSE work in order to proceed with the course as this provides you with the foundation to succeed with AS Maths. For the first few lessons of the AS Mathematics course, these basic algebra skills will be revised. You will sit the test the week beginning the ….. Sept 20…., in order to check your suitability for the course.

HOW TO WORK THROUGH THIS PACKAGE

As a minimum you should read the examples enclosed for each topic and do all the questions marked with a star in the exercises. Do all your work on lined A4 paper, as you will be using a ring binder in the sixth form. You may feel you want to do more questions than this so the answers to all the exercises are included at the back.

If you find any topics particularly difficult, I would recommend using a GCSE revision guide or GCSE Maths website, such as mymaths, BBC bitesize.

> When you arrive in September remember you **must bring in your answers** to all of the starred questions and that you must be ready to sit a test in the week beginning ..13..Sept

If you have any questions, please see M. Hughes

NOTE: the vacation work pack will be handed out during the induction days in the summer

Indices

Key Points

- A number written in the form a^n is an **index number**.
- The **laws of indices** are:

$a^m \times a^n = a^{m+n}$	To multiply two powers of the same number add the indices.
$a^m \div a^n = a^{m-n}$	To divide two powers of the same number subtract the indices.
$(a^m)^n = a^{m \times n}$	To raise a power to a further power multiply the indices together.

You will encounter negative and fractional indices in Chapter 25.

Example Work out a 3^4 b 2^6

a $3^4 = 3 \times 3 \times 3 \times 3$
$= 81$

b $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$
$= 64$

ResultsPlus Watch Out!

Remember that a^3 means that you multiply three as together.
It does not mean $a \times 3$.

Example Write each expression as a power of 5. a $5^6 \times 5^4$ b $5^{12} \div 5^4$ c $(5^3)^2$

a $5^6 \times 5^4 = 5^{4+6}$
$= 5^{10}$ ← Use the index law $a^m \times a^n = a^{m+n}$

b $5^{12} \div 5^4 = 5^{12-4}$
$= 5^8$ ← Use the index law $a^m \div a^n = a^{m-n}$

c $(5^3)^2 = 5^{3 \times 2}$
$= 5^6$ ← Use the index law $(a^m)^n = a^{m \times n}$

Example Work out $\frac{4^7 \times 4}{4^5}$

$\frac{4^7 \times 4}{4^5} = \frac{4^7 \times 4^1}{4^5}$

$= \frac{4^8}{4^5}$ ← Simplify the top of the fraction, add 7 and 1.

$= 4^3$

$= 4 \times 4 \times 4 = 64$ ← As the question asks you to 'Work out', the final answer must be a number.

ResultsPlus Examiner's Tip

'Work out' means 'evaluate' the expression, rather than leaving the answer as a power.

ResultsPlus Watch Out!

Remember that a is the same as a^1.

A

1 Write as a power of a single number

a $6^5 \times 6^7$ b $4^7 \div 4^2$ *c $(7^2)^3$ d $5^9 \div 5^3$ e $3^8 \times 3^2$

2 Work out

a $10^2 \times 10^3$ *b $5^7 \div 5^4$ c $(2^3)^2$ d $3^4 \div 3^2$ *e 4×4^2

3 Find the value of n

a $3^n \div 3^2 = 3^3$ b $8^5 \div 8^n = 8^2$ *c $2^5 \times 2^n = 2^{10}$ d $3^n \times 3^5 = 3^9$ e $2^6 \times 2^3 = 2^n$

4 Write as a power of a single number

a $\frac{3^3 \times 3^5}{3^4}$ b $\frac{5^6 \times 5^7}{5^4}$ *c $\frac{2^8 \times 2^5}{2^7}$ d $\frac{6^{15}}{6 \times 6^9}$ e $\frac{4^2 \times 4^7}{4^3 \times 4^4}$

5 Work out

a $\frac{3^3 \times 3^5}{3^6}$ *b $\frac{2^6 \times 2^2}{2^4}$ c $\frac{4^7}{4 \times 4^5}$ d $\frac{10^5 \times 10^6}{10^7}$ *e $\frac{7^8 \times 7}{7^3 \times 7^4}$

6 Work out the value of n in the following

*a $40 = 5 \times 2^n$ b $32 = 2^n$ c $20 = 2^n \times 5$ *d $48 = 3 \times 2^n$ e $54 = 2 \times 3^n$

Example 4

a Simplify $c^3 \times c^4$

b Simplify $5y^3z^5 \times 2y^2z$

ResultsPlus
Watch Out!

Group like terms together before attempting to use the laws of indices.

a $c^3 \times c^4 = c \times c \times c \times c \times c \times c \times c$

$= c^7$ ← Note: $3 + 4 = 7$.

b $5y^3z^5 \times 2y^2z = 5 \times y^3 \times z^5 \times 2 \times y^2 \times z$ ← z is the same as z^1

$= 5 \times 2 \times y^3 \times y^2 \times z^5 \times z^1$

$= 10 \times y^{3+2} \times z^{5+1}$ ← Using $x^p \times x^q = x^{p+q}$

$= 10 \times y^5 \times z^6$

$= 10y^5z^6$

B

1 Simplify

a $m \times m \times m \times m \times m$

b $2p \times 3p$

c $q \times 4q \times 5q$

2 Simplify

a $a^4 \times a^7$

b $n \times n^3$

c $x^5 \times x$

d $y^2 \times y^3 \times y^4$

3 Simplify

a $2p^2 \times 6p^4$

b $4a \times 3a^4$

c $b^7 \times 5b^2$

d $3n^2 \times 6n$

4 Simplify

a $5t^3u^2 \times 4t^5u^3$

b $2xy^3 \times 3x^5y^4$

c $a^2b^5 \times 7a^3b$

d $4cd^5 \times 2cd^4$

e $2mn^2 \times 3m^3n^2 \times 4m^2n$

Example 5

a Simplify $d^5 \div d^2$

b Simplify $\dfrac{10x^2y^5}{2xy^3}$

ResultsPlus
Examiner's Tip

Write fractions, such as $\dfrac{p^5}{p^3}$ as $p^5 \div p^3$.

a $d^5 \div d^2 = \dfrac{d^5}{d^2} = \dfrac{d \times d \times d \times d \times d}{d \times d}$

$= d^3$ ← Note: $5 - 2 = 3$

b $\dfrac{10x^2y^5}{2xy^3}$ is the same as $10x^2y^5 \div 2xy^3$

$10x^2y^5 \div 2xy^3 = (10 \div 2) \times (x^2 \div x) \times (y^5 \div y^3)$

$= 5 \times x^{2-1} \times y^{5-3}$ ← Using $x^p \div x^q = x^{p-q}$

$= 5 \times x \times y^2$

$= 5xy^2$

C

1 Simplify

a $a^7 \div a^4$

b $b^5 \div b$

c $\dfrac{c^8}{c^5}$

d $d^4 \div d^3$

2 Simplify

a $6q^5 \div 3q^3$

b $12p^7 \div 4p^2$

c $8x^6 \div 2x^5$

d $\dfrac{20y^8}{2y}$

3 Simplify

a $15a^5b^6 \div 3a^3b^2$

b $30p^3q^4 \div 6p^2q$

c $\dfrac{8c^4d^7}{2c^2d^3}$

d $\dfrac{6x^3 \times 2x^4}{4x^2}$

e $\dfrac{5m^2n \times 4mn^2}{2mn^2}$

Example 6 Simplify $(2c^3d)^4$

Method 1

$(2c^3d)^4 = (2)^4 \times (c^3)^4 \times (d)^4$

$= 16 \times c^{3\times4} \times d^{1\times4}$ Using $(x^p)^q = x^{p\times q}$

$= 16 \times c^{12} \times d^4$

$= 16c^{12}d^4$

ResultsPlus
Examiner's Tip
You must apply the power to number terms as well as the algebraic terms.

Method 2

$(2c^3d)^4$ can be written as $2c^3d \times 2c^3d \times 2c^3d \times 2c^3dd$

$= 2 \times 2 \times 2 \times 2 \times c^3 \times c^3 \times c^3 \times c^3 \times d \times d \times d \times d$

$= 16 \times c^{3+3+3+3} \times d^4$ Using $x^p \times x^q = x^{p+q}$

$= 16 \times c^{12} \times d^4$

$= 16c^{12}d^4$

D

1 Simplify
a $(a^7)^2$ *b $(b^3)^5$ c $(c^3)^3$ d $(d^2)^8$

2 Simplify
a $(2p^3)^2$ b $(3q^2)^4$ *c $(5x^4)^2$ *d $\left(\frac{m^4}{2}\right)^3$

3 Simplify
*a $(2x^3y^2)^4$ b $(7e^5f^3)^2$ *c $(5p^5q)^3$ d $\left(\frac{2x^4y^2}{3xy^4}\right)^3$

Key Points

- The laws of indices used so far can be used to develop two further laws.

$x^4 \div x^4 = x^{4-4} = x^0$

Also

$x^4 \div x^4 = 1$ since any term divided by itself is equal to 1.

Therefore $x^0 = 1$

In general

$x^0 = 1$

$x^3 \div x^4$

$= \frac{x \times x \times x}{x \times x \times x \times x} = \frac{1}{x}$

Also, using $x^p \div x^q = x^{p-q}$

$x^3 \div x^4 = x^{3-4} = x^{-1}$

Therefore $x^{-1} = \frac{1}{x}$

In general

$x^{-m} = \frac{1}{x^m}$

The laws of indices can be used further to solve problems with fractional indices.

The square root of x is written $\sqrt{x}$, and you know that:

$\sqrt{x} \times \sqrt{x} = x$

Using $x^p \times x^q = x^{p+q}$

$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$

and so, $x^{\frac{1}{2}} = \sqrt{x}$

Also, $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x$, showing that $x^{\frac{1}{3}} = \sqrt[3]{x}$

In general

$x^{\frac{1}{n}} = \sqrt[n]{x}$

Example Simplify $(3x^4y)^{-2}$

$$(3x^4y)^{-2} = \frac{1}{(3x^4y)^2} \quad \longleftarrow \quad \text{Using } x^{-m} = \frac{1}{x^m}$$

$$= \frac{1}{9x^8y^2} \quad \longleftarrow \quad \text{Using } (x^p)^q = x^{p \times q}$$

ResultsPlus
Examiner's Tip

Remember that a negative power just means 'one over' or 'the reciprocal of'.

E

1 Simplify

a a^{-1} b $(b^2)^{-1}$ c c^{-2} d $(d^3)^{-1}$

2 Simplify

a $(e^3)^{-2}$ b $(f^2)^{-4}$ c $(x^{-1})^{-2}$ d $(y^{-1})^{-1}$

3 Simplify

a $(x^2y^7)^0$ b $(2x^4y^5)^0$ c $(5p^2q^4)^{-1}$ d $(3c^3d)^{-3}$

e $\left(\frac{2p^3q}{3r^2}\right)^{-2}$

Example Simplify $(8x^6y^4)^{\frac{1}{3}}$

$$(8x^6y^4)^{\frac{1}{3}} = 8^{\frac{1}{3}} \times (x^6)^{\frac{1}{3}} \times (y^4)^{\frac{1}{3}} \quad \longleftarrow \quad \text{Using } x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$= \sqrt[3]{8} \times x^{6 \times \frac{1}{3}} \times y^{4 \times \frac{1}{3}} \quad \longleftarrow \quad \text{Using } (x^p)^q = x^{p \times q}$$

$$= 2 \times x^2 \times y^{\frac{4}{3}}$$

$$= 2x^2y^{\frac{4}{3}}$$

ResultsPlus
Examiner's Tip

Remember that the denominator of the index is the root.

F

1 Simplify

a $(9a^4)^{\frac{1}{2}}$ b $(16c^2)^{\frac{1}{4}}$ c $(27e^3f^{-9})^{\frac{1}{3}}$ d $(100x^3y^5)^{\frac{1}{2}}$

2 Simplify

a $(a^4)^{-\frac{1}{2}}$ b $(8c^3)^{-\frac{1}{3}}$ c $(32x^9y^5)^{-\frac{1}{5}}$ d $(x^2y^6)^{-\frac{1}{2}}$

Key Points

- For non-zero values of a
  $a^0 = 1$
- For any number n
  $a^{-n} = \frac{1}{a^n}$

Example Work out the value of a 3^0 b 5^{-1} c 6^{-2} d $\left(\frac{2}{5}\right)^{-2}$

a $3^0 = 1$ ← Any number to the power of zero is 1.

b $5^{-1} = \frac{1}{5}$ ← Use the rule $a^{-n} = \frac{1}{a^n}$

c $6^{-2} = \frac{1}{6^2}$
$= \frac{1}{36}$ ← $6^2 = 6 \times 6 = 36$

d $\left(\frac{2}{5}\right)^{-2} = \frac{1}{\left(\frac{2}{5}\right)^2}$ ← To work out the reciprocal of a fraction, turn the fraction upside down. Square the number on the top and the number on the bottom of the fraction.
$= \left(\frac{5}{2}\right)^2$
$= \frac{25}{4}$

ResultsPlus Examiner's Tip

Do not convert the fraction to a decimal. It is much easier to square the numbers in a fraction than it is to square a decimal.

G

1 Write down the value of these expressions.

* a 7^0	b 8^{-1}	* c 5^{-1}	d 4^0
e $(-2)^{-3}$	* f 9^{-2}	g 10^{-4}	h 145^0
i* $(-3)^{-2}$	j $(-8)^0$	k 16^0	l 10^{-6}

2 Work out the value of these expressions.

* a $\left(\frac{1}{3}\right)^{-1}$	b $\left(\frac{2}{7}\right)^{-1}$	* c $\left(\frac{1}{7}\right)^{-2}$	d $\left(\frac{1}{4}\right)^{-3}$
e $(0.25)^{-2}$	* f $\left(\frac{2}{5}\right)^{-3}$	g $\left(\frac{5}{3}\right)^0$	h $\left(\frac{9}{5}\right)^{-1}$
i* $\left(1\frac{2}{5}\right)^{-2}$	j $\left(1\frac{1}{3}\right)^{-3}$	k $(0.1)^{-4}$	l $(0.2)^{-3}$

Key Points

- Indices can be fractions. In general,
  $a^{\frac{1}{n}} = \sqrt[n]{a}$
- In particular, this means that
  $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{1}{3}} = \sqrt[3]{a}$

Example 11 Find the value of the following

a $25^{\frac{1}{2}}$ b $(-1000)^{\frac{1}{3}}$ c $16^{-0.25}$

a $25^{\frac{1}{2}} = \sqrt{25}$
$= 5$

The square root of 25 is 5 because $5 \times 5 = 25$.

b $(-1000)^{\frac{1}{3}} = \sqrt[3]{-1000}$
$= -10$

The cube root of -1000 is -10 because $-10 \times -10 \times -10 = -1000$.

c $16^{-0.25} = 16^{-\frac{1}{4}}$
$= \frac{1}{16^{\frac{1}{4}}}$
$= \frac{1}{\sqrt[4]{16}}$
$= \frac{1}{2}$

Change the decimal into a fraction $0.25 = \frac{1}{4}$.
Use the rule $a^{-n} = \frac{1}{a^n}$.

$16^{\frac{1}{4}} = \sqrt[4]{16} = 2$ because $2^4 = 16$

Example 12 Work out the value of a $8^{\frac{2}{3}}$ b $16^{-\frac{3}{4}}$

a $8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2$
$= 2^2$
$= 4$

Use the rule $(a^m)^n = a^{mn}$.
Work out the cube root of 8 first.
Then square your answer.

b $16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}}$
$= \frac{1}{(16^{\frac{1}{4}})^3}$
$= \frac{1}{2^3}$
$= \frac{1}{8}$

Use $a^{-n} = \frac{1}{a^n}$.

ResultsPlus
Examiner's Tip
It is easier to work out the root first as this makes the numbers smaller and easier to manage.

H

1 Work out the value of the following.
* a $9^{\frac{1}{2}}$ b $49^{\frac{1}{2}}$ c $100^{\frac{1}{2}}$ d $4^{\frac{1}{2}}$ * e $\left(\frac{1}{4}\right)^{\frac{1}{2}}$

2 Work out the value of
* a $27^{\frac{1}{3}}$ b $1000^{\frac{1}{3}}$ * c $(-64)^{\frac{1}{3}}$ d $125^{\frac{1}{3}}$ * e $\left(\frac{1}{8}\right)^{\frac{1}{3}}$

3 Work out the value of
* a $16^{-\frac{1}{4}}$ b $4^{-\frac{1}{2}}$ c $125^{-\frac{1}{3}}$ * d $\left(\frac{1}{32}\right)^{-\frac{1}{5}}$ * e $\left(\frac{4}{9}\right)^{-\frac{1}{2}}$

4 Work out the value of
* a $27^{\frac{2}{3}}$ b $1000^{\frac{2}{3}}$ * c $64^{\frac{2}{3}}$ * d $16^{\frac{3}{4}}$ e $25^{\frac{3}{2}}$

5 Work out, as a single fraction, the value of
* a $125^{-\frac{2}{3}}$ b $10\,000^{-\frac{3}{4}}$ * c $27^{-\frac{1}{3}}$ * d $8^{-\frac{2}{3}}$ * e $64^{-\frac{3}{2}}$
f $125^{-\frac{2}{3}} \times \left(\frac{1}{5}\right)^2$ g $8^{-\frac{1}{3}} \times \left(\frac{2}{5}\right)^2$

6 Find the value of n.
* a $\frac{1}{8} = 8^n$ * b $64 = 2^n$ c $\frac{1}{\sqrt{5}} = 5^n$ * d $(\sqrt{7})^5 = 7^n$ e $(\sqrt[3]{2})^{11} = 2^n$

Surds

Key Points

- A number written exactly using square roots is called a **surd**.
  $\sqrt{2}$ and $\sqrt{3}$ are both surds.
- $2 - \sqrt{3}$ and $5 + \sqrt{2}$ are examples of numbers written in surd form.
  $\sqrt{4}$ is not a surd as $\sqrt{4} = 2$.
- These two rules can be used to simplify surds.
  $\sqrt{m} \times \sqrt{n} = \sqrt{mn}$ $\quad \dfrac{\sqrt{m}}{\sqrt{n}} = \sqrt{\dfrac{m}{n}}$
- Simplified surds should never have a surd in the denominator.
- To **rationalise the denominator** of a fraction means to get rid of any surds in the denominator.
- To rationalise the denominator of $\dfrac{a}{\sqrt{b}}$ you multiply the fraction by $\dfrac{\sqrt{b}}{\sqrt{b}}$. This ensures that the final fraction has an integer as the denominator.
  $\dfrac{a}{\sqrt{b}} = \dfrac{a}{\sqrt{b}} \times \dfrac{\sqrt{b}}{\sqrt{b}} = \dfrac{a \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \dfrac{a\sqrt{b}}{b}$

Example Simplify $\sqrt{12}$.

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \times 3} \\ &= \sqrt{4} \times \sqrt{3} \\ &= 2\sqrt{3}\end{aligned}$$

Use $\sqrt{m} \times \sqrt{n} = \sqrt{mn}$.
$\sqrt{4} = 2$.

Example Expand and simplify $(2 + \sqrt{3})(4 + \sqrt{3})$.

$$\begin{aligned}(2 + \sqrt{3})(4 + \sqrt{3}) &= 8 + 2\sqrt{3} + 4\sqrt{3} + \sqrt{3} \times \sqrt{3} \\ &= 8 + 6\sqrt{3} + 3 \\ &= 11 + 6\sqrt{3}\end{aligned}$$

Multiply out the brackets.

Simplify the expression.

(I)

1 Find the value of the integer k.
a $\sqrt{8} = k\sqrt{2}$ b $\sqrt{18} = k\sqrt{2}$ c $\sqrt{50} = k\sqrt{2}$ d $\sqrt{80} = k\sqrt{5}$

2 Simplify
a $\sqrt{200}$ b $\sqrt{32}$ c $\sqrt{20}$ d $\sqrt{28}$

3 Solve the equation $x^2 = 30$, leaving your answer in surd form.

4 Expand these expressions. Write your answers in the form $a + b\sqrt{c}$ where a, b and c are integers.
a $\sqrt{3}(2 + \sqrt{3})$ b $(\sqrt{3} + 1)(2 + \sqrt{3})$ c $(\sqrt{5} - 1)(2 + \sqrt{5})$
d $(\sqrt{7} + 1)(2 - \sqrt{7})$ e $(2 - \sqrt{3})^2$ f $(\sqrt{2} + 5)^2$

Example Rationalise the denominator of $\frac{15-\sqrt{5}}{\sqrt{5}}$ and give your answer in the form $a + b\sqrt{5}$.

$$\frac{15-\sqrt{5}}{\sqrt{5}} = \frac{15-\sqrt{5}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{15\sqrt{5} - \sqrt{5}\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}$$
$$= \frac{15\sqrt{5} - 5}{5}$$
$$= -1 + 3\sqrt{5}$$

Simplify the fraction by dividing both parts of the expression on the top of the fraction by 5.

ResultsPlus Watch Out!

Remember to multiply both parts of the expression on the top of the fraction.

J

1 Rationalise the denominators and simplify your answers, if possible.

a $\frac{1}{\sqrt{2}}$ b $\frac{1}{\sqrt{5}}$ c $\frac{5}{\sqrt{10}}$ d $\frac{2}{\sqrt{2}}$ e $\frac{4}{\sqrt{12}}$

2 Rationalise the denominators and give your answers in the form $a + b\sqrt{c}$ where a, b and c are integers.

a $\frac{2+\sqrt{2}}{\sqrt{2}}$ b $\frac{6-\sqrt{2}}{\sqrt{2}}$ c $\frac{10+\sqrt{5}}{\sqrt{5}}$ d $\frac{12-\sqrt{3}}{\sqrt{3}}$ e $\frac{14+\sqrt{7}}{\sqrt{7}}$

Expanding brackets & factorising

Example Expand $20(n+3)$.

Remember to multiply both terms inside the bracket by 20.

$$20(n+3) = 20 \times n + 20 \times 3$$
$$= 20n + 60$$

Write your answer in its simplest form.

Example Expand $3(2x+1)$.

Multiply both $2x$ and 1 by 3.

$$3(2x+1) = 3 \times 2x + 3 \times 1$$
$$= 6x + 3$$

Example Expand $p(p+q-5)$.

$$p(p+q-5) = p \times p + p \times q - p \times 5$$
$$= p^2 + pq - 5p$$

$p \times 5$ is usually written as $5p$.

Example Expand $-2x(3x+1)$.

Multiply both terms by $-2x$.

$$-2x(3x+1) = -2x \times 3x + -2x \times 1$$
$$= -6x^2 - 2x$$

For each term, negative × positive = negative.

K

2 Expand

a $y(y+2)$ b $g(g-3)$ c $2x(x+5)$ d $n(4-n)$
e $a(b+c)$ f $s(3s-4)$ g $3t(2t+1)$ h $4x^2(x-3)$

3 Expand

a $-2(m+3)$ b $-3(2x+2)$ c $-m(m+5)$ d $-4y(2y+3)$
e $-5(p-2)$ f $-3q(1-q)$ g $-2s(s-3)$ h $-3n(4m+n-5)$

Example Expand and simplify $3(2a + 1) + 2(3a + 5)$.

$$3(2a + 1) + 2(3a + 5) = 6a + 3 + 6a + 10$$

Expand each bracket separately.

$$= 12a + 13$$

Collect like terms.

Example Expand and simplify $3x(y - 2) - 2y(x - 3)$.

$$3x(y - 2) - 2y(x - 3) = 3xy - 6x - 2xy + 6y$$
$$= xy - 6x + 6y$$

For the last term, negative × negative = positive.

L

1 Expand and simplify

a $3(t - 1) + 5t$ b $6p + 3(p + 2)$ c $6(w + 1) + 5w$

d $3(d + 2) + 4(d - 2)$ e $3a + b + 2(a + b)$ f $2(5x - y) + 5(y - x + 1)$

2 Expand and simplify

a $3(y + 10) - 2(y + 5)$ b $6(2a + 1) - 3(a + 4)$ c $x - 5(x + 3)$

d $q(q + 3) - 3(q + 1)$ e $2n(n - 2) - n(2n + 1)$ f $3m(2 + 5m) - 4m(1 + m)$

3 Expand and simplify

a $5(t - 4) - 4(t - 1)$ b $3(x + 3) - 2(x - 5)$ c $2g(g + 1) - g(g + 1)$

d $6c(2c - 3) - c(4 - c)$ e $4s(s + 3) - 2(1 - s)$ f $p(p + q) - q(p - q)$

4 Expand and simplify

a $7s - 4(s + 1)$ b $12m + 3(m + 2)$ c $8f^2 - 3f(f + 1)$

d $5n + n(n - 1)$ e $2x - x(x - y)$ f $7p - 2p(1 - p)$

Example Factorise $12b + 8$.

$$12b + 8 = 4(\quad)$$
$$= 4(3b + 2)$$

The common factor of $12b$ and 8 is 4.
Note that you would not usually write the $4(\quad)$ but it is there to remind you to find the common factor first.

Check this multiplies out to give $12b + 8$.

Example Factorise $2 - 6y$.

$$2 - 6y = 2(\quad)$$
$$= 2(1 - 3y)$$

Pick out the common factor first.
1 is needed as the first term in the bracket.

Example Factorise $x^2 + 3x$.

$$x^2 + 3x = x(\quad)$$
$$= x(x + 3)$$

The common factor of x^2 and $3x$ is x.
Remember to check by multiplying out.

Example Factorise $15p - 10q - 20pq$.

$$15p - 10q - 20pq = 5(\quad)$$
$$= 5(3p - 2q - 4pq)$$

Find the common factor of all three terms.

M_1

1 Factorise

a $3x + 6$ b $2y - 2$ * c $5p + 10q$ d $14t - 7$
e $8s + 2t$ f $9a + 18b$ * g $15u + 5v + 10w$ h $xt - yt$
* i $ac - c$ * j $6x^2 + 9x + 3$ * k $2p^2 - 2p$ * l $q^2 - q$
m $4x^2 + 3x$ * n $2h - 5h^2$ o $p^3 + 2p$ * p $s^2 + s^3$

2 Factorise completely

* a $5xy + 5xt$ b $3ad - 6ac$ c $6pq + 4hp$ * d $8xy - 4y$
e $4pq + 2ps + 8pt$ * f $mn - kmn$ g $2x^2 + 4x$ h $12s^2 - 24s$
* i $6f^2 + 2f^3$ j $y^4 + y^2$ k $3cd^2 - 5c^2d$ l $a^3b + ab^3$
m $8pqr + 10prs$ n $14a^2b - 7ab^2 + 21ab$ o $15x^2y - 35x^2y^2$ p $(3y)^2 + 3y$

Example 13 Factorise $5(x + 2)^2 - 3(x + 2)$.

$$\begin{aligned}5(x + 2)^2 - 3(x + 2) &= (x + 2)[\quad]\\ &= (x + 2)[5(x + 2) - 3]\\ &= (x + 2)[5x + 10 - 3]\\ &= (x + 2)(5x + 7)\end{aligned}$$

$(x + 2)$ is a common factor.

Simplify the expression inside the square bracket.

M_2

1 Factorise

* a $(x + 3)^2 + 2(x + 3)$ b $x(x - y) + y(x - y)$ * c $p(p + 4) - 3p$
d $(2t + s)(2t - s) + (2t - s)$ * e $(a - 5)^2 - 2(a - 5)$ f $(2d + 1)^2 + (2d + 1)$

Example 15 Expand and simplify $(x + 2)(x + 3)$.

Method 1

$$\begin{aligned}(x + 2)(x + 3) &= x(x + 3) + 2(x + 3)\\ &= x^2 + 3x + 2x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

Take each term in the first bracket, in turn, and multiply it by the second bracket.
Expand the brackets.
Collect the like terms.

* You may use the foil or grid method to multiply out brackets too.

Example 16 Expand and simplify $(m + 2)^2$.

$$\begin{aligned}(m + 2)^2 &= (m + 2)(m + 2)\\ &= m(m + 2) + 2(m + 2)\\ &= m^2 + 2m + 2m + 4\\ &= m^2 + 4m + 4\end{aligned}$$

Write out $(m + 2)^2$ in full.

ResultsPlus
Watch Out!
Note that $(a + b)^2$ is not equal to $a^2 + b^2$.

Example 17 Expand and simplify $(2t - 1)(3t - 2)$.

Method 1

$$\begin{aligned}(2t - 1)(3t - 2) &= 2t(3t - 2) - 1(3t - 2)\\ &= 6t^2 - 4t - 3t + 2\\ &= 6t^2 - 7t + 2\end{aligned}$$

Check your signs are correct.

1 Expand and simplify

a $(x + 3)(x + 4)$ b $(x + 1)(x + 2)$ c $(x + 2)(x - 5)$
d $(y - 2)(y + 3)$ e $(y + 1)(y - 2)$ f $(x - 2)(x - 3)$
g $(a - 4)(a - 5)$ h $(x + 2)^2$ i $(p + 4)^2$
j $(k - 7)^2$ k $(a + b)^2$ l $(a - b)^2$

2 Expand and simplify

a $(x + 1)(2x + 1)$ b $(x - 1)(3x + 1)$ c $(2x + 3)(x + 4)$
d $(y - 3)(3y + 1)$ e $(2p + 1)(p + 3)$ f $(2t + 1)(3t + 2)$
g $(3s + 2)(2s + 5)$ h $(2x - 3)(2x + 5)$ i $(3y + 2)(4y - 1)$
j $(2a - 1)(3a - 2)$ k $(3x + 2)^2$ l $(2k - 1)^2$

3 Expand and simplify

a $(x + y)(x + 2y)$ b $(x - y)(x + 2y)$ c $(x + y)(x - 2y)$
d $(x - y)(x - 2y)$ e $(2p + 3q)(3p - q)$ f $(3s - 2t)(2s - t)$
g $(2a + 3b)^2$ h $(2a - 3b)^2$

Key Points

- Factorising is the reverse process to expanding brackets so, for example, factorising $x^2 + 5x + 6$ gives $(x + 2)(x + 3)$.
- To factorise the quadratic expression $x^2 + bx + c$
  - find two numbers whose product is $+c$ and whose sum is $+b$
  - use these two numbers, p and q, to write down the factorised form $(x + p)(x + q)$.
- To factorise the quadratic expression $ax^2 + bx + c$
  - work out the value of ac
  - find a pair of numbers whose product is $+ac$ and sum is $+b$
  - rewrite the x term in the expression using these two numbers
  - factorise the first two terms and the last two terms
  - pick out the common factor and write as the product of two brackets.
- Any expression which may be written in the form $a^2 - b^2$, known as the difference of two squares, can be factorised using the result $a^2 - b^2 = (a + b)(a - b)$.

Example 11

Factorise $x^2 + 7x + 12$.

The pairs of numbers whose product is 12 are:

$+1 \times +12$ -1×-12
$+2 \times +6$ -2×-6
$+3 \times +4$ -3×-4

$+3 \times +4 = +12$ ← Find two numbers whose product is $+12$ and whose sum is $+7$.
$+3 + +4 = +7$
$x^2 + 7x + 12 = (x + 3)(x + 4)$ ← Put into factorised form using the numbers $+3$ and $+4$.

ResultsPlus Examiner's Tip

You may find it helpful to start by writing down all the pairs of numbers whose product is $+12$.

Example 13 Factorise $x^2 - 10x + 25$.

The pairs of numbers whose product is $+25$ are:

$+1 \times +25 \qquad -1 \times -25$
$+5 \times +5 \qquad -5 \times -5$

$-5 \times -5 = +25$
$-5 + -5 = -10$
$x^2 - 10x + 25 = (x - 5)(x - 5)$

This may also be written as $(x - 5)^2$.

ResultsPlus
Examiner's Tip
You can check your answer by expanding the brackets.

(O)

2 Factorise

a $x^2 + 8x + 15$	✗ b $x^2 + 8x + 7$	✗ c $x^2 + 9x + 20$
d $x^2 + 6x + 9$	✗ e $x^2 - 6x + 5$	✗ f $x^2 - 2x + 1$
g $x^2 + 3x - 18$	✗ h $x^2 - 3x - 18$	i $x^2 + 3x - 28$
j $x^2 - x - 12$	✗ k $x^2 + 2x - 24$	✗ l $x^2 - 4$
✗ m $x^2 - 81$		

Example 21 Factorise $x^2 - 100$.

$x^2 - 100 = x^2 - 10^2$
$= (x + 10)(x - 10)$

Substitute $a = x$ and $b = 10$ into $a^2 - b^2 = (a + b)(a - b)$.

ResultsPlus
Examiner's Tip
It will help you in the examination if you learn $a^2 - b^2 = (a + b)(a - b)$.

Example 22
a Factorise $p^2 - q^2$.
b Hence, without using a calculator, find the value of $101^2 - 99^2$.

a $p^2 - q^2 = (p + q)(p - q)$
b $101^2 - 99^2$
$= (101 + 99)(101 - 99)$
$= 200 \times 2$
$= 400$

Use the result $a^2 - b^2 = (a + b)(a - b)$.
Substitute $p = 101$ and $q = 99$ in the answer to part (a).

Work out each bracket.

(P)

1 Factorise

✗ a $x^2 - 36$	b $x^2 - 49$	✗ c $y^2 - 144$
✗ d $25 - y^2$	✗ e $w^2 - 2500$	✗ f $10\,000 - a^2$

Example Factorise $3x^2 - 7x + 4$.

$a = +3, b = -7, c = +4$
$ac = 12, b = -7$

Find two numbers whose product is $+12$ and whose sum is -7.

$-3 \times -4 = +12$
$-3 + -4 = -7$

Replace $-7x$ with $-3x - 4x$.

$3x^2 - 7x + 4 = 3x^2 - 3x - 4x + 4$

Factorise by grouping.

$= 3x(x - 1) - 4(x - 1)$

Pick out the common factor and write as the product of two brackets.

$= (x - 1)(3x - 4)$

$3x^2 - 7x + 4 = (x - 1)(3x - 4)$

Q

1 Factorise

* a $5x^2 + 16x + 3$ * b $2x^2 + 11x + 5$ * c $3x^2 + 4x + 1$ d $8x^2 + 6x + 1$
e $6x^2 + 13x + 6$ * f $6x^2 - 7x + 1$ * g $5x^2 - 7x + 2$ h $12x^2 - 11x + 2$
i $8x^2 + 2x - 3$ j $2x^2 - 7x - 15$ k $7x^2 - 19x - 6$ l $3x^2 - 10x - 8$
* m $4y^2 + 12y + 5$ n $6y^2 - 13y + 2$ o $6y^2 - 25y + 25$

Solving Linear Equations

Example Solve $4(x + 1) = 11$

$4(x + 1) = 11$
$4x + 4 = 11$

Expand the left-hand side by multiplying out the brackets.

$4x + 4 - 4 = 11 - 4$

Solve the equation as in Section 13.1.

$4x = 7$

$\frac{4x}{4} = \frac{7}{4}$

$x = \frac{7}{4}$

$x = 1\frac{3}{4}$ or 1.75

ResultsPlus
Examiner's Tip
Your answer can be written as a fraction or as a decimal.

R Solve

* **1** $2(x + 3) = 12$ **2** $5(y + 4) = 35$ * **3** $4(x - 1) = 5$ **4** $3(2y - 1) = 9$

* **5** $13 = 4(x + 3)$ **6** $2(1 - w) = 10$ **7** $5(3 - 4z) = 20$ * **8** $2 = 3(1 + 3x)$

* **9** $2(2x + 3) + 1 = 11$ * **10** $17 = 6 - 3(5 - 2y)$

Example 3 Solve $5x + 5 = 3 - 3x$

$$5x + 5 = 3 - 3x$$
$$5x + 5 + 3x = 3 - 3x + 3x$$

Add $3x$ to both sides of the equation. Remember $-3x + 3x = 0$.

$$8x + 5 = 3$$
$$8x + 5 - 5 = 3 - 5$$
$$8x = -2$$
$$\frac{8x}{8} = \frac{-2}{8}$$
$$x = -\tfrac{1}{4} \text{ or } -0.25$$

Solve the equation as in Section 13.1.

> **ResultsPlus**
> **Watch Out!**
> Always show each stage of your working.

Example Solve $3 - 6x = 7 - 3x$

Here both terms in x have a negative coefficient.

$$3 - 6x = 7 - 3x$$
$$3 - 6x + 6x = 7 - 3x + 6x$$

Add $6x$ to both sides of the equation.

$$3 = 7 + 3x$$
$$3 - 7 = 3x$$
$$-4 = 3x$$
$$x = -\frac{4}{3}$$

> **ResultsPlus**
> **Examiner's Tip**
> Collect the terms in x on the side of the equation that gives them a positive coefficient.

5 Solve

1 $4a + 3 = 8 + 2a$ **2** $5b + 3 = b - 7$ **3** $3c - 2 = 5c - 8$ **4** $d + 7 = 5d + 15$

5 $3 - 2e = 4 - 3e$ **6** $1 - 7f = 3f + 10$ **7** $2(x - 4) = x + 7$

8 $2x + 5 = 1 + 3(2 + x)$ **9** $3(4x + 1) + 2(1 - 5x) = 2 + x$

10 $6 + 2(x - 3) = x - 3(1 - 2x)$

Example Solve $\dfrac{12}{p + 2} = 3$

$$\frac{12}{p + 2} = 3$$
$$\frac{12}{\cancel{p + 2}} \times \cancel{(p + 2)} = 3 \times (p + 2)$$

Multiply both sides of the equation by $(p + 2)$. The terms in $(p + 2)$ on the left-hand side cancel out.

$$12 = 3(p + 2)$$
$$12 = 3p + 6$$
$$12 - 6 = 3p$$
$$3p = 6$$
$$p = 2$$

> **ResultsPlus**
> **Examiner's Tip**
> Always try to remove the fraction first.

Example Solve $\frac{x+1}{2} - \frac{4x-1}{3} = \frac{5}{12}$

$$\frac{x+1}{2} - \frac{4x-1}{3} = \frac{5}{12}$$

$$12 \times \frac{x+1}{2} - 12 \times \frac{4x-1}{3} = 12 \times \frac{5}{12}$$ ← Multiply each of the three terms by 12.

$${}^{6}\cancel{12} \times \frac{x+1}{\cancel{2}_1} - {}^{4}\cancel{12} \times \frac{4x-1}{\cancel{3}_1} = {}^{1}\cancel{12} \times \frac{5}{\cancel{12}_1}$$

$$6(x+1) - 4(4x-1) = 5$$
$$6x + 6 - 16x + 4 = 5$$ Remember: $-4 \times -1 = +4$
$$-10x + 10 = 5$$
$$-10x = -5$$
$$x = \frac{1}{2}$$

T

Solve

1. $\frac{p}{5} + 3 = 7$
2. $\frac{q+2}{3} = 4$
3. $\frac{m}{2} + \frac{m}{5} = 21$
4. $\frac{x}{6} + 1 = \frac{x-4}{4}$
5. $3(2y - 10) = \frac{4y-7}{2}$
6. $2\left(\frac{x}{3} - 3\right) = 16$
7. $\frac{1}{2n} + \frac{1}{3n} = 7$
8. $\frac{3t+6}{10} + \frac{5-2t}{5} = 6$
9. $2 - \frac{1-x}{3} = \frac{5x+2}{9}$
10. $\frac{3y-4}{2} - \frac{2y+1}{5} = \frac{1-y}{3}$

Solving Linear inequalities

Key Points

- You solve a linear inequality using a similar method to the one you use for solving a linear equation
- If you multiply both sides of an inequality by a negative number, then you must reverse the inequality sign.

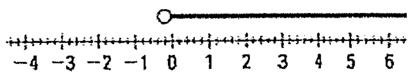
Example 3 Solve $3(x + 2) > 5 - x$ and show your answer on a number line.

$3(x+2) > 5 - x$ ← Expand the brackets.
$3x + 6 > 5 - x$

$3x + 6 + x > 5$ ← Add x to both sides.
$4x + 6 > 5$

$4x > 5 - 6$ ← Subtract 6 from both sides.
$4x > -1$

$x > -0.25$ ← Divide both sides by 4.



Example 4 Solve $-3x \leqslant 12$.

$-3x \leqslant 12$

$-12 \leqslant 3x$ ← Add $3x$ to both sides and subtract 12 from both sides.

$3x \geqslant -12$

$x \geqslant -4$ ← Divide both sides by 3.

U

1 Solve these inequalities and show each answer on a number line.

* a $x + 1 > 5$ b $x - 3 \leqslant -2$ * c $2x + 5 \leqslant 1$ d $10x - 7 > 9$

2 Solve these inequalities.

a $3x < x + 9$ * b $5x - 3 > 2x + 9$ * c $2(x + 3) \leqslant 11$ * d $5x - 7 > 3(x + 2)$

3 Solve these inequalities.

* a $x + 3 \geqslant 5(x - 2)$ * b $3(x + 1) < 4(x - 5)$

* c $\frac{2 - 3x}{5} \leqslant 1 + \frac{x}{2}$ d $\frac{5x - 3}{4} + 1 \geqslant \frac{1 - 2x}{6}$

Example 5 $-3 \leqslant 2p - 1 < 8$

p is an integer. Find all the possible values of p.

$-3 \leqslant 2p - 1$ and $2p - 1 < 8$ ← Write the two inequalities seperately.

$1 - 3 \leqslant 2p$ $2p < 8 + 1$ ← Solve each inequality.

$-2 \leqslant 2p$ $2p < 9$

$-1 \leqslant p$ $p < 4.5$

So $-1 \leqslant p < 4.5$

$p = -1, 0, 1, 2, 3, 4.$ ← Write down the integer values satisfying the inequality.

V Find the possible integer values of x in these inequalities.

* **1** a $-2 < x \leqslant 5$ b $-5 < x < 2$ c $0 < x \leqslant 3$ d $-5 \leqslant x \leqslant 4$

* **2** a $-8 < 2x \leqslant 6$ b $-21 < 5x < 36$ * c $-5 < 10x \leqslant 42$ d $-11 \leqslant 3x \leqslant 28$

* **3** a $-5 < 2x + 1 < 9$ * b $-7 < 3x - 2 \leqslant 11$ c $-12 < 4x - 7 \leqslant 10$ d $-9 \leqslant 2x + 5 \leqslant 13$

Solving Simultaneous Linear equations

Example 1

Solve the simultaneous equations

$4x - y = 3$

$x + y = 7$

Method 1

$4x - y = 3 \quad (1)$

$x + y = 7 \quad (2)$

Label the equations (1) and (2).

$5x + 0 = 10$

Since $-y$ and $+y$ are of different sign, add equations (1) and (2) to eliminate terms in y.

$5x = 10$ so $x = 2$

Divide both sides by 5.

When $x = 2$, $2 + y = 7$

$y = 7 - 2 = 5$

Substitute $x = 2$ into equation (2) and solve to find the value of y.

So the solution is $x = 2$, $y = 5$.

Check: $4 \times 2 - 5 = 8 - 5 = 3$ ✓

Check your solution by substituting into equation (1).

ResultsPlus Examiner's Tip

When deciding which unknown to eliminate, if possible choose the unknown where the signs are different. You can then eliminate the unknown by adding the equations.

Method 2

$4x - y = 3 \quad (1)$

$x + y = 7 \quad (2)$

Label the equations (1) and (2).

$y = 7 - x$

Rearrange equation (2) to make y the subject.

$4y - (7 - x) = 3$

Substitute $y = 7 - x$ into equation (1).

$4x - 7 + x = 3$

$5x - 7 = 3$

$5x = 3 + 7 = 10$

Expand the bracket and solve by the **balance method**.

$5x = 10$ so $x = 2$

Divide both sides by 5.

When $x = 2$, $2 + y = 7$

$y = 7 - 2 = 5$

Substitute $x = 2$ into equation (2) and solve to find the value of y.

So the solution is $x = 2$, $y = 5$.

Check: $4 \times 2 - 5 = 8 - 5 = 3$ ✓

Check your solution by substituting into equation (1).

ResultsPlus Examiner's Tip

If a fraction is introduced when making y the subject of an equation, use an alternative method since the fraction will complicate your working.

Example 2

Solve the simultaneous equations

$5x - 6y = 13$

$3x - 4y = 8$

$$5x - 6y = 13 \quad (1)$$
$$3x - 4y = 8 \quad (2)$$
$$15x - 18y = 39 \quad (3)$$
$$15x - 20y = 40 \quad (4)$$

Multiply (1) by 3 and (2) by 5 to make the coefficients of x equal. Label the new equations (3) and (4).

$$0 + 2y = -1$$

Subtract equation (4) from equation (3) to eliminate the terms in x. $-18y - (-20y) = -18y + 20y = +2y$

$$y = -\tfrac{1}{2}$$
$$5x - (6 \times -\tfrac{1}{2}) = 13$$

Substitute $y = -\frac{1}{2}$ into equation (1).

$$5x - (-3) = 13$$
$$5x + 3 = 13$$
$$5x = 10$$
$$x = 2$$

So the solution is $x = 2$, $y = -\frac{1}{2}$.

Check: $3 \times 2 - (4 \times -\frac{1}{2}) = 6 + 2 = 8$

Check your solution by substituting into equation (2).

W

Solve these simultaneous equations.

✗ **1** $2x + y = 9$, $x + y = 5$

2 $3x - y = 12$, $2x + y = 13$

✗ **3** $5x - 2y = 9$, $3x - 2y = 7$

4 $x + 4y = 6$, $3x - 2y = 4$

✗ **5** $x + 2y = 9$, $y = x + 3$

6 $2x + 5y = 12$, $y = 3 - x$

7 $5x - y = -4$, $y = 2x + 1$

8 $3x - 4y = -2$, $y = x + 1$

9 $8x - 3y = -2$, $y = 3 - 2x$

✗ **10** $4x - 3y = 14$, $2x + 2y = -7$

✗ **11** $3x + 2y = 11$, $2x - 5y = 1$

✗ **12** $4x + 6y = 5$, $3x + 4y = 4$

13 $5x + 4y = 5$, $3x - 5y = -34$

✗ **14** $7x - 2y = 13$, $4x - 3y = 13$

✗ **15** $4x - 3y = 5$, $2x + 2y = -1$

Quadratic Equations

Factorising

Example 5 Solve

a $2x^2 = 6x$ b $y^2 - y - 20 = 0$

ResultsPlus
Watch Out!

When there is a power of x on both sides of an equation, do not simply divide both sides by one of the powers of x because the solution $x = 0$ may be lost.

a $2x^2 = 6x$

$2x^2 - 6x = 0$ ← Rearrange into the form $ax^2 + bx + c = 0$.

$2x(x - 6) = 0$ ← Factorise.

So either $2x = 0$ or $(x - 6) = 0$ ← Solve the linear equations.
giving the two solutions $x = 0$ and $x = 6$.

b $y^2 - y - 20 = 0$ ← Factorise into two bracketed terms.

$(y - 5)(y + 4) = 0$

So either $(y - 5) = 0$ or $(y + 4) = 0$. ← Remember: You are looking for two numbers whose product is -20 and whose sum is -1 (i.e. -5 and $+4$).

The two solutions are $y = 5$ and $y = -4$.

Example 6 Solve $q(q + 4) + 4 = 6q + 3$.

$q(q + 4) + 4 = 6q + 3$

$q^2 + 4q + 4 = 6q + 3$ ← Expand the brackets and rearrange into the form $ax^2 + bx + c = 0$.

$q^2 + 4q + 4 - 6q - 3 = 0$

$q^2 - 2q + 1 = 0$

$(q - 1)(q - 1) = 0$ ← Factorise.

So either $q - 1 = 0$ or $q - 1 = 0$, giving
the two equal solutions $q = 1$ and $q = 1$. ← The solutions are both the same.
We say the solution is $q = 1$.

Example 7 Solve $4x^2 - 25 = 0$.

Method 1

$4x^2 - 25 = 0$

$4x^2 = 25$

$x^2 = 25 \div 4 = 6.25$ ← Take the square root of both sides.

$x = \pm\sqrt{6.25}$

So the two solutions are $x = 2.5$ or $x = -2.5$.

Method 2

$4x^2 - 25 = 0$

$(2x - 5)(2x + 5) = 0$ ← Factorise by the difference of two squares method (see Section 9.4).

So either $(2x - 5) = 0$ or $(2x + 5) = 0$.

So the two solutions are $x = 2.5$ or $x = -2.5$.

1 Solve

a $x(x-4)=0$ b $(a+5)(a-3)=0$ c $(2m-1)(4m-9)=0$

d $y^2+2y=0$ e $t^2-t=0$ f $4p^2-7p=0$

2 Solve

a $x^2-6x+8=0$ b $x^2+7x+6=0$ c $x^2+x-12=0$

d $x^2-6x+9=0$ e $x^2-5x-36=0$ f $x^2-16=0$

g $x^2+10x+25=0$ h $x^2-100=0$

3 Solve

a $5x^2+26x+5=0$ b $3x^2-11x+6=0$ c $2x^2+7x-4=0$ d $5x^2+14x-3=0$

Completing the square

Example 4 Write x^2+4x+5 in the form $(x+p)^2+q$, stating the values of p and q.

$x^2+4x=(x+2)^2-4$ ← Ignore the constant term. Find the perfect square which will give the correct terms in x^2 and x, then subtract 4 to make the identity true.

So

$x^2+4x+5=(x+2)^2-4+5$ ← Add 5 to obtain x^2+4x+5.

$=(x+2)^2+1$ ← Simplify the expression.

$p=2, q=1$ ← Compare $(x+2)^2+1$ with $(x+p)^2+q$ and write down the values of p and q.

Example 5

a Write the expression $2y^2-12y-5$ in the form $p(y+q)^2+r$.

b Hence write down the minimum possible value of $2y^2-12y-5$.

c Find the value of y for which $2y^2-12y-5$ has its minimum value.

a $2y^2-12y-5=2(y^2-6y)-5$ ← Take out the coefficient of y^2 for the y^2 and y terms. Leave the constant term separate.

$=2[(y-3)^2-9]-5$ ← Complete the square for y^2-6y.

$=2(y-3)^2-18-5$ ← Multiply out the square brackets.

$=2(y-3)^2-23$ ← Simplify the expression so that it is in the required form.

b The minimum possible value of any square number is zero so $(y-3)^2 \geqslant 0$ for any value of y.

The minimum possible value of $2(y-3)^2-23=2\times 0-23$. ← Substitute $(y-3)^2=0$ into the answer to part a.

The minimum value of $2y^2-12y-5$ is therefore -23.

c The minimum value of $2y^2-12y-5$ occurs when $y=3$. ← Find the value of y which makes $(y-3)^2=0$.

4

1 Write the following in the form $(x+p)^2+q$.

a x^2+4x b x^2+10x c x^2+12x d x^2-2x
e x^2-14x f x^2-24x g x^2+x h x^2-3x
i x^2+4x+7 j $x^2+8x+17$ k $x^2+10x-20$ l $x^2-6x+11$
m $x^2-20x+80$ n $x^2-26x-1$ o x^2-x+1 p x^2+5x-5

2 Write the following in the form $a(x+p)^2+q$.

a $2x^2+12x$ b $2x^2-4x+5$
c $3x^2-12x+10$ d $5x^2+50x+100$

Solving quadratics by completing the square

Example

Solve $x^2-12x+9=0$.
Give your solutions: a in surd form
b correct to 3 significant figures.

$x^2-12x+9$ will not factorise into two brackets since no two integers have a product of 9 and a sum of -12.

$x^2-12x+9=0$
$(x-6)^2-36+9=0$
$(x-6)^2-27=0$

Complete the square for x^2-12x. Comparing this with $p(x+q)^2+r=0$ gives, $p=1$, $q=-6$ and $r=-27$.

$(x-6)^2=27$
$x-6=\pm\sqrt{27}$ — Take the square root of both sides.
$x-6=\pm3\sqrt{3}$ — Add 6 to both sides.
$x=6\pm3\sqrt{3}$

a The two solutions are $x=6+3\sqrt{3}$ and $x=6-3\sqrt{3}$.
b The two solutions are $x=11.2$ and $x=0.804$.

2

1 Solve these quadratic equations, giving your solutions in surd form.

a $x^2-6x-2=0$ b $x^2+4x+1=0$ c $x^2+10x-12=0$
d $x^2-2x-7=0$ e $2x^2-6x-3=0$ f $5x^2+12x+3=0$

2 Solve these quadratic equations, giving your solutions correct to 2 decimal places.

a $x^2+8x+5=0$ b $x^2-9x+6=0$ c $x^2+x-8=0$
d $2x^2+4x-5=0$ e $6x^2-3x-2=0$ f $10x^2-5x-4=0$

Quadratic formula – you need to learn this

quadratic formula.

$ax^2+bx+c=0$

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$ LEARN

If the value of b^2-4ac is negative, the quadratic equation does not have any real solutions.

Example

Solve $x^2 - 5x + 3 = 0$.
Give your solutions correct to 2 decimal places.

$x^2 - 5x + 3 = 0$
$a = 1, b = -5, c = 3$

Compare with $ax^2 + bx + c = 0$ and write down the values of a, b and c.

$$x = -(-5) \pm \frac{\sqrt{(-5)^2 - 4 \times 1 \times 3}}{2 \times 1}$$

Substitute a, b and c into the quadratic formula.

$$x = \frac{5 \pm \sqrt{25 - 12}}{2}$$

$$x = \frac{5 + \sqrt{13}}{2}$$

$$\text{or } x = \frac{5 - \sqrt{13}}{2}$$

The solutions are $x = 4.30$ or $x = 0.70$

ResultsPlus
Examiner's Tip

In equations like $x^2 + bx + c = 0$ it is helpful to write it as $1x^2 + bx + c = 0$ so that the value of a is clearly 1.

Example

Solve $5x^2 + x - 3 = 0$.
Give your solutions correct to 2 decimal places.

$5x^2 + x - 3 = 0$
$a = 5, b = 1, c = -3$

Compare with $ax^2 + bx + c = 0$ and write down the values of a, b and c.

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 5 \times -3}}{2 \times 5}$$

Substitute a, b and c into the quadratic formula.

$$x = \frac{-1 \pm \sqrt{1 + 60}}{10}$$

$$x = \frac{-1 + \sqrt{61}}{10} \text{ or } x = \frac{-1 - \sqrt{61}}{10}$$

The solutions are $x = 0.68$ or $x = -0.88$.

AA

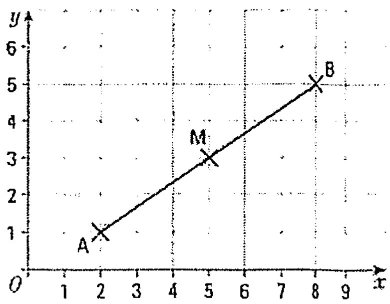
Solve these quadratic equations. Give your solutions correct to 3 significant figures.

1 $x^2 + 4x + 2 = 0$	**2** $x^2 + 7x + 5 = 0$	**3** $x^2 + 6x - 4 = 0$
4 $x^2 + x - 10 = 0$	**5** $x^2 - 4x - 7 = 0$	**6** $x^2 - 5x + 3 = 0$
7 $2x^2 + 4x + 1 = 0$	**8** $5x^2 - 9x + 2 = 0$	**9** $6x^2 - 5x - 8 = 0$
10 $10x^2 + 3x - 2 = 0$	**11** $4x^2 - 7x + 2 = 0$	**12** $x(x - 1) = x + 5$

Co-ordinate Geometry

Key Points

- A line joining two points is called a line segment. AB is the line segment joining points A and B.
- The **midpoint** of a line is halfway along the line.
- To find the midpoint you should add the x-coordinates and divide by 2, and add the y-coordinates and divide by 2.
- The midpoint of the line segment AB between A (x_1, y_1) and B (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. LEARN



Example Work out the coordinates of the midpoint of the line segment PQ where P is (2, 3) and Q is (7, 11).

x-coordinate $2 + 7 = 9$
$9 \div 2 = 4\frac{1}{2}$ ← Add the x-coordinates and divide by 2.

y-coordinate $3 + 11 = 14$
$14 \div 2 = 7$ ← Add the y-coordinates and divide by 2.

The midpoint is $(4\frac{1}{2}, 7)$.

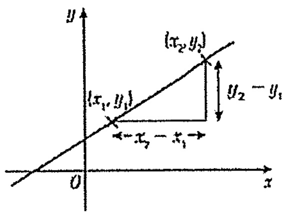
AB

3 Work out the coordinates of the midpoint of each of these line segments.

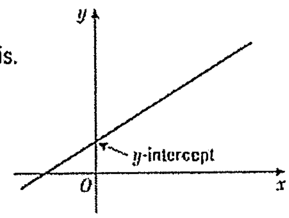
* a AB when A is $(-1, -1)$ and B is $(9, 9)$
b PQ when P is $(2, -4)$ and Q is $(-6, 9)$
* c ST when S is $(5, -8)$ and T is $(-2, 1)$
d CD when C is $(1, 7)$ and D is $(-7, 2)$
* e UV when U is $(-2, 3)$ and V is $(6, -8)$
f GH when G is $(-2, -6)$ and H is $(7, 3)$

Key Points LEARN ALL OF THIS

- The **gradient** of a straight line is a measure of its slope.
- Steeper lines have larger gradients.
- Gently sloping lines have smaller gradients.
- Gradient of a line $= \dfrac{\text{change in } y\text{-direction}}{\text{change in } x\text{-direction}}$
- Lines which slope upwards from left to right have positive gradients.
- Lines which slope downwards from left to right have negative gradients.
- The gradient of the line through the points (x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

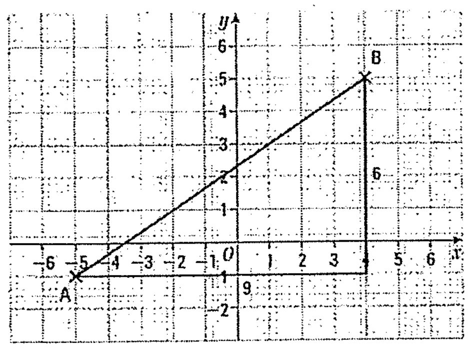


- The y**-intercept** of a line is the value of y when $x = 0$.
It is shown by the point where the graph crosses the y-axis.



Example Find the gradient of the line joining the points A $(-5, -1)$ and B $(4, 5)$.

Method 1



← Draw a diagram to help you.

$$\frac{\text{vertical distance}}{\text{horizontal distance}} = \frac{\text{difference in } y\text{-coordinates}}{\text{difference in } x\text{-coordinates}}$$

$$= \frac{5 + 1}{4 + 5}$$

$$= \frac{6}{9}$$

$$\text{Gradient} = \frac{2}{3}$$

Method 2

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{5 - (-1)}{4 - (-5)}$$

$$= \frac{6}{9}$$

$$\text{Gradient} = \frac{2}{3}$$

Use $(x_1, y_1) = (-5, -1)$ and $(x_2, y_2) = (4, 5)$. Put $x_1 = -5, y_1 = -1$ and $x_2 = 4, y_2 = 5$ into the formula for m.

AC

1 Work out the gradient of the line joining these pairs of points:

* **a** $(4, 2), (6, 3)$ **b** $(-1, 3), (5, 4)$

* **c** $(-4, 5), (1, 2)$ **d** $(2, -3), (6, 5)$

* **e** $(-3, 4), (7, -6)$ **f** $(-12, 3), (-2, 8)$

Key Points

- The straight line with equation $y = mx + c$ has gradient m.
- The straight line with equation $y = mx + c$ crosses the y-axis at the point $(0, c)$.
- The point $(0, c)$ is known as the y-intercept.

Example 4

For the lines with equations

a $y = 5x + 4$

b $3x + 2y = 6$

find:

i the gradient of the line

ii the y-intercept of the line.

a $y = 5x + 4$ ← Compare $y = 5x + 4$ with $y = mx + c$.

i gradient $= 5$ ← Write down the value of the gradient m from the term in x.

ii y-intercept $= 4$ ← Write down the value of the y-intercept c from the constant term.

b i $3x + 2y = 6$ ← Rearrange the equation $3x + 2y = 6$ into the form $y = mx + c$.

$2y = 6 - 3x$ ← Subtract $3x$ from both sides.

$y = 3 - 1.5x$ ← Divide both sides by 2.

$y = -1.5x + 3$

gradient $= -1.5$

ii y-intercept $= 3$

AD

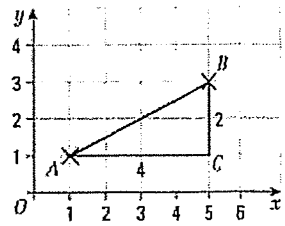
2 Find i the gradient and ii the y-intercept of the lines with the equations

* a $y = 4x + 1$ b $y = 3x - 4$ * c $y = \frac{2}{3}x + 4$

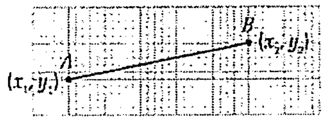
* d $2x + 5y = 20$ * e $4x - 3y = 12$ * f $x - 2y = 0$

Key Points

- By creating a right-angled triangle, Pythagoras' Theorem can be used to find the length between two points on a line.



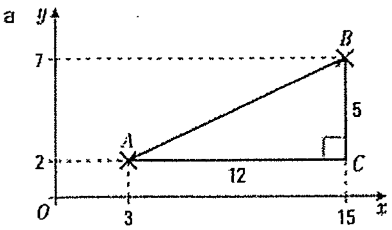
- The length of the line segment AB between $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Example 1

Find the length of the line joining a A (3, 2) and B (15, 7) b P (−9, 4) and Q (7, −5)

a



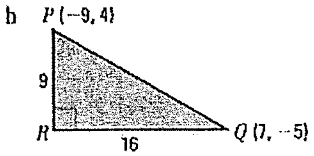
AC = 15 − 3 = 12
BC = 7 − 2 = 5

Draw a sketch showing A and B and complete the right-angled triangle ABC.

$AB^2 = 12^2 + 5^2$
$AB^2 = 144 + 25 = 169$
$AB = \sqrt{169} = 13$

Use Pythagoras' Theorem to find the length of AB.

b



QR = 7 − −9 = 7 + 9 = 16
PR = 4 − −5 = 4 + 5 = 9

Draw a sketch showing P and Q and complete the right-angled triangle PQR.

$PQ^2 = 16^2 + 9^2$
$PQ^2 = 256 + 81 = 337$
$PQ = \sqrt{337} = 18.4$ (to 3 s.f.)

Use Pythagoras' Theorem to find the length of PQ.

AE

1 Work out the length of the line joining each of these pairs of points.

a (3, 1) and (11, 7)
b (2, 5) and (12, 29)
c (−6, 9) and (8, 13)
d (−4, −6) and (6, 12)
e (9, −15) and (−11, 6)
f (0, −5) and (9, −11)

Key Point

- If a line has gradient m then any line drawn parallel to it also has gradient m and any line drawn **perpendicular** to it has gradient $-\frac{1}{m}$ (the negative reciprocal of m).

Example 2

Find the equation of the line parallel to $y = 3x + 7$ and passing through (0, −2).

$y = 3x + 7$, gradient $m = 3$, y-intercept is (0, 7)

Compare $y = 3x + 7$ with $y = mx + c$.

The gradient of any line parallel to $y = 3x + 7$ is 3
so the equation of any line parallel to $y = 3x + 7$ is $y = 3x + c$.

Parallel lines have equal gradients.

The required line has y-intercept (0, −2).
The equation is $y = 3x - 2$.

Write down the value of c from the coordinates of the y-intercept given.

Example 15 Find the equation of any line which is perpendicular to $y = 2x - 9$.

$y = 2x - 9$, gradient $m = 2$

The gradient of any line perpendicular to this has gradient $-\frac{1}{2}$. ← Find the negative reciprocal of 2.

The equation of any line with gradient $-\frac{1}{2}$ is of the form $y = -\frac{1}{2}x + c$. ← Use $y = mx + c$.

So $y = -\frac{1}{2}x + 1$ is one example of a line perpendicular to the line $y = 2x - 9$. ← Pick any value for c.

AF

1 Copy and complete the following table to show the gradients of pairs of lines l_1 and l_2 which are perpendicular to each other.

	a	b	c	d	e
Gradient of line l_1	3	-4	$\frac{1}{5}$		
Gradient of line l_2				3	$-\frac{1}{6}$

2 Write down the equation of a line parallel to the line with the equation

a $y = 2x + 5$ b $y = \frac{1}{3}x - 1$ c $y = 4 - x$

3 Write down the equation of a line perpendicular to the line with the equation

a $y = x - 6$ b $y = 3x + 2$ c $y = 1 - \frac{1}{2}x$

AG

Proof

Key point

A **proof** is a logical argument for a mathematical statement. To prove a statement is true, you must show that it will be true in *all* cases.

To prove a statement is not true you can find a **counter-example** – an example that does not fit the statement.

9 a **Communication / Reasoning** Prove that the sum of any odd number and any even number is always odd.

b **Reasoning** Explain why any odd number can be written as $2n + 1$ or $2n - 1$.

Q9a hint Let $2n$ be any even number. Let $2n + 1$ be any odd number.

10 **Communication / Reasoning**

a The nth even number is $2n$. Explain why the next even number is $2n + 2$.

b Prove that the product of two consecutive even numbers is a multiple of 4.

11 **Communication / Reasoning** Prove that the product of any two odd numbers is odd.

Simultaneous equations – one linear, one quadratic

Example

Solve these simultaneous equations.

① $2x + y = 3$

② $x^2 + y = 6$

$y = 3 - 2x$ — Rearrange equation ① to make y the subject.

$x^2 + (3 - 2x) = 6$ — Substitute $y = 3 - 2x$ into equation ②

$x^2 - 2x + 3 = 6$

$x^2 - 2x - 3 = 0$ — Expand the bracket and rearrange so the right-hand side is 0.

$(x + 1)(x - 3) = 0$ — Solve the quadratic equation.

So either $(x + 1) = 0$ or $(x - 3) = 0$

$x = -1$ or $x = 3$

$2 \times (-1) + y = 3$ — Substitute $x = -1$ into equation ① to find one value of y.

$-2 + y = 3$

$y = 5$

$2 \times 3 + y = 3$ — Substitute $x = 3$ into equation ① to find the second value of y.

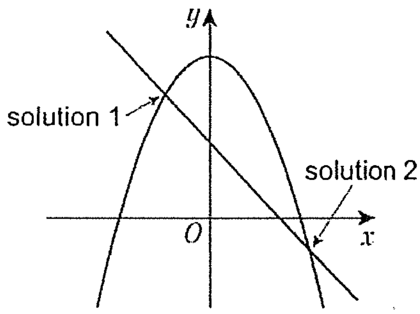
$6 + y = 3$

$y = -3$

So the solutions are $x = -1, y = 5$ and $x = 3, y = -3$

Key point

A pair of quadratic and linear simultaneous equations can have two possible solutions.



AH

4 Solve these simultaneous equations.

a $y = x$
$x^2 + y = 12$

b $2x - y = 7$
$x^2 - 15 = y$

c $y - 4x = 6$
$y = 2x^2 + 3x + 5$

d $y = 5x - 3$
$y = 3x^2 + 6x - 7$

e $x^2 + y^2 = 4$
$3x + 5 = y$

5 Solve these simultaneous equations. Give your answers correct to 2 decimal places where appropriate.

a $y + 3x = 8$
$y = x^2 + 2x + 4$

b $2y - 4x = 6$
$y = x^2 + x - 5$

AI

Quadratic Inequalities

Key point

To solve a quadratic inequality:
- Solve as a quadratic equation
- Sketch the graph
- Use the graph to find the values that satisfy the inequality.

14 Find the set of values that satisfy each inequality.

* a $x^2 - 2x - 3 < 0$ * b $x^2 + 3x - 10 < 0$ * c $x^2 + 5x + 4 > 0$
d $x^2 + 7x + 10 < 0$ e $x^2 - 6x + 8 > 0$ f $x^2 - 6x + 5 < 0$

Q14a hint Solve $x^2 - 2x - 3 = 0$ first.

Turning points

Key point

To find the coordinate of the turning point, write the equation in completed square form:
$y = a(x + b)^2 + c$
$(x + b)^2 \geqslant 0$, so the minimum for y is when $x + b = 0$ and $y = c$

Example

a Does the graph of $y = x^2 + 8x + 15$ have a maximum or a minimum point?
b Find the coordinates of the turning point.

a Minimum

The coefficient of x^2 is positive, so the turning point is a minimum.

b $y = x^2 + 8x + 15$
$y = (x + 4)^2 - 16 + 15$
$y = (x + 4)^2 - 1$
$x + 4 = 0$, so $x = -4$
Minimum at $(-4, -1)$

Write the quadratic function in completed square form.

The smallest value that y can take is -1. This occurs when $(x + 4)^2 = 0$. $(x + 4)^2$ cannot be less than 0 because a square is always positive.
Solve the equation to find the x-coordinate.

AJ

9 **Reasoning** For each quadratic function, work out the coordinates of the turning point and state whether it is a maximum or a minimum.

* a $y = x^2 - 2x + 4$ * b $y = -x^2 - 6x - 11$
* c $y = x^2 - 10x + 23$ * d $y = 2x^2 + 12x + 13$
e $y = 3x^2 - 12x + 13$ f $y = -2x^2 - 4x + 2$

Discussion What do you notice about the completed square form and the coordinates of the turning point?

Q9b hint $y = -(x^2 + 6x + 11)$
$= -((x + 3)^2 + 2)$
$= -(x + 3)^2 - 2$
y-coordinate of the turning point

Algebraic fractions

Example

Simplify fully $\dfrac{x^2 + 5x + 4}{x^2 - 3x - 28}$

$$\frac{x^2 + 5x + 4}{x^2 - 3x - 28} = \frac{(x+1)(x+4)}{(x-7)(x+4)}$$

Factorise the numerator and denominator.

$$= \frac{x+1}{x-7}$$

Divide the numerator and denominator by the common factor $(x + 4)$.

AL

8 Simplify fully

a $\dfrac{x^2 + 8x + 15}{x^2 + 2x - 15}$ b $\dfrac{x^2 - 11x + 30}{x^2 + x - 42}$ c $\dfrac{x^2 - 25}{(x+5)^2}$

Q8c hint Factorise $(x^2 - 25)$ using the difference of two squares.

9 **Exam-style question**

Simplify fully

$\dfrac{x^2 + 14x + 49}{x^2 - 49}$

(3 marks)

Exam hint
First factorise the numerator and denominator.
Use the fact that $a^2 - b^2 = (a + b)(a - b)$.

10 Simplify fully

a $\dfrac{2x^2 - x - 3}{3x^2 + x - 2}$ b $\dfrac{5x^2 + 14x - 3}{6x^2 + 23x + 15}$ c $\dfrac{25x^2 - 1}{25x^2 + 10x + 1}$

11 **Exam-style question**

Simplify fully

$\dfrac{x^2 + 3x - 4}{2x^2 - 5x + 3}$

(3 marks)

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Exam hint
1 mark is awarded for correctly factorising the numerator; 1 mark for factorising the denominator; and 1 mark for the correct final answer.

Expanding triple brackets

AL

3 Expand the expression $(x^2 + 4x + 1)(x + 2)$

Q3 hint Multiply each term in the first bracket by each term in the second bracket. Then simplify.

$(x^2 + 4x + 1)(x + 2)$

4 Copy and complete to expand the expression

$(x + 2)(x + 4)(x + 3) = (x^2 + \square x + \square)(x + 3)$

$= x^3 + \square + x^2 + \square x = \square$

5 Expand the expressions

a $(x + 2)(x + 5)(x + 1)$ b $(x - 3)(x + 4)(x - 2)$

c $(x + 2)(x - 1)(x + 3)$ d $x(x + 5)(x - 4)$

e $(x + 1)^2(x - 1)$ f $(x + 3)^3$

Functions

Key point 5

A function is a rule for working out values of y for given values of x.
For example, $y = 3x$ and $y = x^2$ are functions. The notation $f(x)$ is read as 'f of x'. f is the function.
$f(x) = 3x$ means the function of x is $3x$.

AM

4 $f(x) = \frac{10}{x}$. Work out

a $f(5)$ b $f(-2)$
c $f(\frac{1}{2})$ d $f(-20)$

Q4a hint Substitute $x = 5$ into $\frac{10}{x}$.

5 **Reasoning** $h(x) = 5x^2$. Alice says that $h(2) = 100$.

a Explain what Alice did wrong. b Work out $h(2)$.

6 $g(x) = 2x^3$. Work out

a $g(3)$ b $g(-1)$
c $g(\frac{1}{2})$ d $g(-5)$

Q6 hint Use the priority of operations.

8 $g(x) = 5x - 3$. Work out the value of a when

a $g(a) = 12$ b $g(a) = 0$ c $g(a) = -7$

Q8a hint $g(a) = 5a - 3 = 12$
Solve for a.

9 $f(x) = x^2 - 8$. Work out the values of a when

a $f(a) = 17$ b $f(a) = -4$
c $f(a) = 0$ d $f(a) = 12$

Q9c hint Write your answer as a surd in its simplest form.

Key point 6

fg is a composite function. To work out $fg(x)$, first work out $g(x)$ and then substitute your answer into $f(x)$.

13 **Reasoning** $f(x) = 6 - 2x$, $g(x) = x^2 + 7$. Work out

a $gf(2)$ b $gf(7)$
c $fg(4)$ d $fg(5)$

Q13a hint First work out $f(2)$ and then substitute your answer into $g(x)$.

14 **Reasoning** $f(x) = 4x - 3$, $g(x) = 10 - x$, $h(x) = x^2 + 7$. Work out

a $gf(x)$ b $fg(x)$
c $fh(x)$ d $hf(x)$
e $gh(x)$ f $hg(x)$

Q14a hint $gf(x)$ means substitute $f(x)$ for x in $g(x)$.
$gf(x) = g(4x - 3) = 10 - (4x - 3) =$ ______

Key point 7

The inverse function reverses the effect of the original function.

Example 7

Find the inverse function of $x \to 5x - 1$

Communication hint $x \to 5x - 1$ is another way of showing $f(x) = 5x - 1$

$x \to \boxed{\times 5} \to \boxed{-1} \to 5x - 1$

Write the function as a function machine.

$\frac{x+1}{5} \leftarrow \boxed{\div 5} \leftarrow \boxed{+1} \leftarrow x$

Reverse the function machine to find the inverse function. Start with x as the input.

The inverse function of $x \to 5x - 1$ is $x \to \frac{x+1}{5}$

15 Find the inverse of each function.

a $x \to 4x + 9$
b $x \to \frac{x}{3} - 4$
c $x \to 2(x + 6)$
d $x \to 7(x - 4) - 1$

Q15a hint You can check your answer by substituting e.g. $x = 2$ into the original function and then the answer into the inverse.

Q15d hint Simplify the function first.
$x \to 7(x - 4) - 1$ is the same as $x \to 7x - 29$

Answers

A

1 a 6^{12} b 4^5 c 7^6
 d 5^6 e 3^{10}
2 a 100 000 b 125 c 64
 d 9 e 64
3 a 5 b 3 c 5
 d 4 e 9
4 a 3^4 b 5^9 c 2^6
 d 6^5 e 4^2
5 a 9 b 16 c 16
 d 10 000 e 49
6 a 3 b 5 c 2
 d 4 e 3

B

1 a m^5 b $6p^2$ c $20q^3$
2 a a^{11} b n^4 c x^6 d y^9
3 a $12p^6$ b $12a^5$ c $5b^9$ d $18n^3$
4 a $20t^8u^5$ b $6x^6y^7$ c $7a^5b^6$ d $8c^2d^9$
 e $24m^6n^5$

C

1 a a^3 b b^4 c c^3 d d
2 a $2q^2$ b $3p^5$ c $4x$ d $10y^7$
3 a $5a^2b^4$ b $5pq^3$ c $4c^2d^4$ d $3x^5$
 e $10m^2n$

D

1 a a^{14} b b^{15} c c^9 d d^{16}
2 a $4p^6$ b $81q^8$ c $25x^8$ d $\frac{m^{12}}{8}$
3 a $16x^{12}y^8$ b $49e^{10}f^6$ c $125p^{15}q^3$ d $\frac{8x^9}{27y^6}$

E

1 a $\frac{1}{a}$ b $\frac{1}{b^2}$ c $\frac{1}{c^2}$ d $\frac{1}{d^3}$
2 a $\frac{1}{e^6}$ b $\frac{1}{f^8}$ c x^2 d y
3 a 1 b 1 c $\frac{1}{5p^2q^4}$
 d $\frac{1}{27c^9d^3}$ e $\frac{9r^4}{4p^6q^2}$

F

1 a $3a^2$ b $2c^{\frac{1}{2}}$ c $\frac{3e}{f^3}$ d $10x^{\frac{3}{2}}y^{\frac{5}{2}}$
2 a $\frac{1}{a^2}$ b $\frac{1}{2c}$ c $\frac{1}{2x^{\frac{9}{5}}y}$ d $\frac{1}{x^{\frac{1}{2}}y^{\frac{3}{2}}}$

G

1 a 1 b $\frac{1}{8}$ c $\frac{1}{5}$ d 1
 e $-\frac{1}{8}$ f $\frac{1}{81}$ g $\frac{1}{10\,000}$ h 1
 i $\frac{1}{9}$ j 1 k 1 l $\frac{1}{1\,000\,000}$
2 a 3 b $\frac{7}{2}$ c 49 d 64
 e 16 f $15\frac{5}{8}$ g 1 h $\frac{5}{9}$
 i $\frac{25}{49}$ j $\frac{27}{64}$ k 10 000 l 125

H

1 a 3 b 7 c 10
 d 2 e $\frac{1}{2}$
2 a 3 b 10 c -4
 d 5 e $\frac{1}{2}$
3 a $\frac{1}{2}$ b $\frac{1}{2}$ c $\frac{1}{5}$
 d 2 e $\frac{3}{2}$
4 a 9 b 100 c 16
 d 8 e 125
5 a $\frac{1}{25}$ b $\frac{1}{1000}$ c $\frac{1}{3}$
 d $\frac{1}{4}$ e $\frac{1}{512}$ f $\frac{1}{625}$
 g $\frac{2}{25}$
6 a $n=-1$ b $n=6$ c $n=-\frac{1}{2}$
 d $n=\frac{5}{2}$ e $n=\frac{11}{3}$

I

1 a 2 b 3 c 5 d 4
2 a $10\sqrt{2}$ b $4\sqrt{2}$ c $2\sqrt{5}$ d $2\sqrt{7}$
3 $x=\pm\sqrt{30}$
4 a $3+2\sqrt{3}$ b $5+3\sqrt{3}$ c $3+\sqrt{5}$
 d $-5+\sqrt{7}$ e $7-4\sqrt{3}$ f $27+10\sqrt{2}$

J

1 a $\frac{\sqrt{2}}{2}$ b $\frac{\sqrt{5}}{5}$ c $\frac{\sqrt{10}}{2}$
 d $\sqrt{2}$ e $\frac{2\sqrt{3}}{3}$
2 a $1+\sqrt{2}$ b $-1+3\sqrt{2}$ c $1+2\sqrt{5}$
 d $-1+4\sqrt{3}$ e $1+2\sqrt{7}$

K

2 a y^2+2y b g^2-3g c $2x^2+10x$
 d $4n-n^2$ e $ab+ac$ f $3s^2-4s$
 g $6t^2+3t$ h $4x^3-12x^2$
3 a $-2m-6$ b $-6x-6$ c $-m^2-5m$
 d $-8y^2-12y$ e $-5p+10$ f $-3q+3q^2$
 g $-2s^2+6s$ h $-12mn-3n^2+15n$

L

1 a $8t-3$ b $9p+6$ c $11w+6$
 d $7d-2$ e $5a+3b$ f $5x+3y+5$
2 a $y+20$ b $9a-6$ c $-4x-15$
 d q^2-3 e $-5n$ f $11m^2+2m$
3 a $t-16$ b $x+19$ c g^2+g
 d $13c^2-22c$ e $4s^2+14s-2$ f p^2+q^2
4 a $3s-4$ b $3m+18$ c $5f^2-3f$
 d n^2+4n e $2x-x^2+xy$ f $2p^2+5p$

M₁

1 a $3(x+2)$ b $2(y-1)$ c $5(p+2q)$
 d $7(2t-1)$ e $2(4s+t)$ f $9(a+2b)$
 g $5(3u+v+2w)$ h $t(x-y)$
 i $c(a-1)$ j $3(2x^2+3x+1)$
 k $2p(p-1)$ l $q(q-1)$ m $x(4x+3)$
 n $h(2-5h)$ o $p(p^2+2)$ p $s^2(1+s)$
2 a $5x(y+t)$ b $3a(d-2c)$ c $2p(3q+2h)$
 d $4y(2x-1)$ e $2p(2q+s+4t)$
 f $mn(1-k)$ g $2x(x+2)$ h $12s(s-2)$
 i $2f^2(3+f)$ j $y^2(y^2+1)$ k $cd(3d-5c)$
 l $ab(a^2+b^2)$ m $2pr(4q+5s)$ n $7ab(2a-b+3)$
 o $5x^2y(3-7y)$ p $3y(3y+1)$

M₂

1 a $(x+3)(x+5)$ b $(x-y)(x+y)$
 c $p(p+1)$ d $(2t-s)(2t+s+1)$
 e $(a-5)(a-7)$ f $2(d+1)(d+1)$

N

1 a $x^2+7x+12$ b x^2+3x+2
 c $x^2-3x-10$ d y^2+y-6
 e y^2-y-2 f x^2-5x+6
 g $a^2-9a+20$ h x^2+4x+4
 i $p^2+8p+16$ j $k^2-14k+49$
 k $a^2+2ab+b^2$ l $a^2-2ab+b^2$
2 a $2x^2+3x+1$ b $3x^2-2x-1$
 c $2x^2+11x+12$ d $3y^2-8y-3$
 e $2p^2+7p+3$ f $6t^2+7t+2$
 g $6s^2+19s+10$ h $4x^2+4x-15$
 i $12y^2+5y-2$ j $6a^2-7a+2$
 k $9x^2+12x+4$ l $4k^2-4k+1$
3 a $x^2+3xy+2y^2$ b $x^2+xy-2y^2$
 c $x^2-xy-2y^2$ d $x^2-3xy+2y^2$
 e $6p^2+7pq-3q^2$ f $6s^2-7st+2t^2$
 g $4a^2+12ab+9b^2$ h $4a^2-12ab+9b^2$

O

2 a $(x+3)(x+5)$ b $(x+1)(x+7)$ c $(x+4)(x+5)$
 d $(x-5)(x-1)$ e $(x-8)(x-1)$ f $(x-1)^2$
 g $(x-3)(x+6)$ h $(x-6)(x+3)$ i $(x-4)(x+7)$
 j $(x-4)(x+3)$ k $(x-4)(x+6)$ l $(x-2)(x+2)$
 m $(x-9)(x+9)$

P

1 a $(x-6)(x+6)$ b $(x-7)(x+7)$
c $(y-12)(y+12)$ d $(5-y)(5+y)$
e $(w-50)(w+50)$ f $(100-a)(100+a)$

Q

1 a $(5x+1)(x+3)$ b $(2x+1)(x+5)$
c $(3x+1)(x+1)$ d $(4x+1)(2x+1)$
e $(3x+2)(2x+3)$ f $(6x-1)(x-1)$
g $(5x-2)(x-1)$ h $(4x-1)(3x-2)$
i $(4x+3)(2x-1)$ j $(2x+3)(x-5)$
k $(7x+2)(x-3)$ l $(3x+2)(x-4)$
m $(2y+1)(2y+5)$ n $(6y-1)(y-2)$
o $(3y-5)(2y-5)$

R

1 $x=3$ 2 $y=3$
3 $x=2.25$ 4 $y=2$
5 $x=0.25$ 6 $w=-4$
7 $z=-0.25$ 8 $x=-\frac{1}{9}$
9 $x=1$ 10 $y=\frac{13}{3}$

S

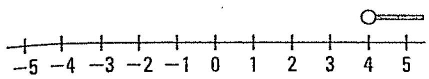
1 $a=2.5$ 2 $b=-2.5$
3 $c=3$ 4 $d=-2$
5 $e=1$ 6 $f=-\frac{9}{10}=0.9$
7 $x=15$ 8 $x=-2$
9 $x=-3$ 10 $x=\frac{3}{5}=0.6$

T

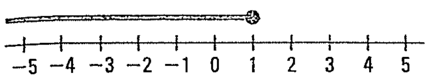
1 $p=20$ 2 $q=10$
3 $m=30$ 4 $x=24$
5 $y=\frac{53}{8}$ 6 $x=33$
7 $n=\frac{5}{42}$ 8 $t=-44$
9 $x=6.5$ 10 $y=\frac{76}{43}$

U

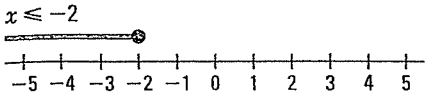
1 a $x>4$



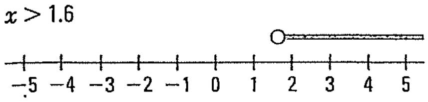
b $x\leqslant 1$



c $x\leqslant -2$



d $x>1.6$



2 a $x<4.5$ b $x>4$ c $x\leqslant 2.5$ d $x>6.5$
3 a $x\leqslant 3.25$ b $x>23$ c $x\geqslant -\frac{6}{11}$ d $x\geqslant -\frac{1}{19}$

V

1 a $-1, 0, 1, 2, 3, 4, 5$
b $-4, -3, -2, -1, 0, 1.$
c $1, 2, 3$
d $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4$
2 a $-3, -2, -1, 0, 1, 2, 3$
b $-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7$
c $0, 1, 2, 3, 4$
d $-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$
3 a $-2, -1, 0, 1, 2, 3$
b $-1, 0, 1, 2, 3, 4$
c $-1, 0, 1, 2, 3, 4$
d $-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4$

W

1 $x=4, y=1$ 2 $x=5, y=3$
3 $x=1, y=-2$ 4 $x=2, y=1$
5 $x=1, y=4$ 6 $x=1, y=2$
7 $x=-1, y=-1$ 8 $x=-2, y=-1$
9 $x=0.5, y=2$ 10 $x=0.5, y=-4$
11 $x=3, y=1$ 12 $x=2, y=-0.5$
13 $x=-3, y=5$ 14 $x=1, y=-3$
15 $x=0.5, y=-1$

X

1 a $0, 4$ b $3, -5$ c $0.5, 2.25$
d $0, -2$ e $0, 1$ f $0, 1.75$
2 a $2, 4$ b $-1, -6$ c $3, -4$
d 3 e $9, -4$ f $4, -4$
g -5 h $10, -10$
3 a $-5, -\frac{1}{5}$ b $3, \frac{2}{3}$ c $\frac{1}{2}, -4$
d $-3, \frac{1}{5}$

Y

1 a $(x+2)^2-4$ b $(x+5)^2-25$
c $(x+6)^2-36$ d $(x-1)^2-1$
e $(x-7)^2-49$ f $(x-12)^2-144$
g $(x+0.5)^2-0.25$ h $(x-1.5)^2-2.25$
i $(x+2)^2+3$ j $(x+4)^2+1$
k $(x+5)^2-45$ l $(x-3)^2+2$
m $(x-10)^2-20$ n $(x-13)^2-170$
o $(x-0.5)^2+1.25$ p $(x+2.5)^2-11.25$
2 a $2(x+3)^2-18$ b $2(x-1)^2+3$
c $3(x-2)^2-2$ d $5(x+5)^2-25$

Z

1 a $x=3\pm\sqrt{11}$ b $x=-2\pm\sqrt{3}$
c $x=-5\pm\sqrt{37}$ d $x=1\pm 2\sqrt{2}$
e $x=\dfrac{3\pm\sqrt{15}}{2}$ f $x=\dfrac{-6\pm\sqrt{21}}{5}$
2 a $x=-0.68, x=-7.32$ b $x=8.27, x=0.73$
c $x=2.37, x=-3.37$ d $x=0.87, x=-2.87$
e $x=0.88, x=-0.38$ f $x=0.93, x=-0.43$

AA

1 $-0.586, -3.41$ 2 $-0.807, -6.19$
3 $0.606, -6.61$ 4 $2.70, -3.70$
5 $5.32, -1.32$ 6 $4.30, 0.697$
7 $-0.293, -1.71$ 8 $1.54, 0.260$
9 $1.64, -0.811$ 10 $0.322, -0.622$
11 $1.39, 0.360$ 12 $3.45, -1.45$

AB

1 a $(4, 4)$ b $(-2, 2.5)$ c $(1.5, -3.5)$
d $(-3, 4.5)$ e $(2, -2.5)$ f $(2.5, -1.5)$

AC

1 a $\frac{1}{2}$ **b** $\frac{1}{6}$
c $-\frac{3}{5}$ **d** 2
e -1 **f** $\frac{1}{2}$

AD

a i 4 ii 1
b i 3 ii -4
c i $\frac{2}{3}$ ii 4
d i -0.4 ii $(0, 4)$
e i $1\frac{1}{3}$ ii $(0, -4)$
f i $\frac{1}{2}$ ii $(0, 0)$

AE

1 a 10 b 26 c $\sqrt{212}=14.6$
d $\sqrt{424}=20.6$ e 29 f $\sqrt{117}=10.8$

AF

1 a $-\frac{1}{3}$ b $\frac{1}{4}$ c -5 d $-\frac{1}{3}$ e 6

2 a $y = 2x + c$ for any value of c except 5
b $y = \frac{1}{3}x + c$ for any value of c except -1
c $y = c - x$ for any value of c except 4

3 a $y = c - x$ for any value of c
b $y = c - \frac{1}{3}x$ for any value of c
c $y = 2x + c$ for any value of c

AG

9 $2n + 1 + 2n = 4n + 1 =$ odd.

10 a The next even number will be two more (because the next number, which is one more, will be odd).
b $(2n)(2n + 2) = 4n^2 + 4n = 4(n^2 + n)$. This is divisible by 4.

11 $(2n + 1)(2n - 1) = 4n^2 - 1$. $4n^2$ must be even, so $4n^2 - 1$

AH

4 a $x = -4, y = -4$ or $x = 3, y = 3$
b $x = -2, y = -11$ or $x = 4, y = 1$
c $x = -0.5, y = 4$ or $x = 1, y = 10$
d $x = -1.33, y = -9.67$ or $x = 1, y = 2$
e $x = -1.89, y = -0.66$ or $x = -1.11, y = 1.66$

5 a $x = 0.70, y = 5.90$ or $x = -5.70, y = 25.10$
b $x = 3.37, y = 9.74$ or $x = -2.37, y = -1.74$

AI

14 a $\{x : -1 < x < 3\}$
b $\{x : -5 < x < 2\}$
c $\{x : x < -4\} \cup \{x : x > -1\}$

AJ

9 a Minimum (1, 3)
b Maximum (−3, −2)
c Minimum (5, −2)
d Minimum (−3, −5)
e Minimum (2, 1)
f Maximum (−1, 4)

AK

8 a $\frac{x+3}{x-3}$ b $\frac{x-5}{x+7}$ c $\frac{x-5}{x+5}$

9 $\frac{x+7}{x-7}$

10 a $\frac{2x-3}{3x-2}$ b $\frac{5x-1}{6x+5}$ c $\frac{5x-1}{5x+1}$

11 $\frac{x+4}{2x-3}$

AL

3 $x^3 + 6x^2 + 9x + 2$

4 $x^3 + 9x^2 + 26x + 24$

5 a $x^3 + 8x^2 + 17x + 10$ b $x^3 - x^2 - 14x + 24$
c $x^3 + 4x^2 + x - 6$ d $x^3 + x^2 - 20x$
e $x^3 + x^2 - x - 1$ f $x^3 + 9x^2 + 27x + 27$

AM

4 a 2 b −5 c 20 d $-\frac{1}{2}$

5 a Alice first multiplied 5 by 2 to get 10. Then she worked out 10 squared, which is 100.
b 20

6 a 54 b −2 c $\frac{1}{4}$ d −250

7 a 5 b 56 c 480 d 2.5
e 600 f −33

8 a $a = 3$ b $a = \frac{3}{5}$ c $a = -\frac{4}{5}$

9 a $a = \pm 5$ b $a = \pm 2$ c $a = \pm 2\sqrt{2}$ d $a = \pm 2\sqrt{5}$

10 a $a = 0, a = -3$ b $a = 1, a = -5$
c $a = -1, a = -2$ d $a = -1, a = -3$

11 a $5x + 1$ b $5x - 13$ c $10x - 8$ d $35x - 28$
e $10x - 4$ f $20x - 4$

12 a $3x^2 + 3$ b $6x^2 - 8$ c $12x^2 - 4$ d $3x^2 - 4$

13 a 11 b 71 c −40 d −58

14 a $-4x + 13$ b $37 - 4x$
c $4x^2 + 25$ d $16x^2 - 24x + 16$
e $-x^2 + 3$ f $107 - 20x + x^2$

15 a $x \to \frac{x-9}{4}$ b $x \to 3(x + 4)$
c $x \to \frac{x}{2} - 6$ d $x \to \frac{x+1}{7} + 4$